Network Representation of Electromagnetic Fields and Forces Using Generalized Bond Graphs

by G. D. NIJEN TWILHAAR

Twente University of Technology, P.O. Box 217, 7500 AE Enschede, The Netherlands

ABSTRACT: *We show that it is possible to describe electromagnetic (E-M) fields with a generalized network representation (generalized bond graphs). E-M fields in moving matter, forces due to E-M fields (Lorentz force, etc.) and field transformations are included in the network description. The relations of these E-M phenomena with respect to each other are clearly represented by the bond graph. We also show that it is not possible to describe E-M phenomena in moving matter with conventional bond graphs, but that a generalized bond graph concept is required.*

The description of simple E-M devices with conventional bond graphs is based on rather drastic assumptions, i.e. quasi-static conditions (E-M radiation neglected), homogeneous fields, isotropic linear material, etc. These assumptions are not made in this paper.

Nomenclature

$\bar{A}$	magnetic vector potential [Wb m^{-1}]
$\bar{B}$	magnetic induction [Wb m^{-2}]
c	velocity of light [m s^{-1}]
$\bar{D}$	electric displacement [C m^{-2}]
e_j	intensive variable (domain j)
$\bar{E}$	electric field intensity [V m^{-1}]
$\bar{F}$	force [N]
$\bar{H}$	magnetic field intensity [A m^{-1}]
$\bar{I}$	vector current [A]
$\bar{j}$	current density [A m^{-2}]
L	length [m]
$\bar{m}$	vector magnetomotive force [A]
$\bar{p}$	mechanical momentum [Ns]
P	pressure [Pa]
q_j	extensive variable (domain j)
$\bar{Q}$	vector charge displacement [C]
R_1	radius [m]
R	electric resistance [V A^{-1}]
S	surface area [m^2]
$\bar{S}$	Poynting vector [W m^{-2}]
$\bar{U}$	vector voltage [V]
$\bar{v}$	velocity [m s^{-1}]
V	volume [m^3]

W energy [J]
ε electric permittivity [$CV^{-1}\,m^{-1}$]
μ magnetic permeability [$Wb\,A^{-1}\,m^{-1}$]
ρ charge density [$C\,m^{-3}$]
σ specific conductivity [$A\,V^{-1}\,m^{-1}$]
ϕ electric potential [V]
$\bar{\Phi}$ vector flux [Wb]

Sub- and superscripts
U_i ith component of the vector $\bar{U}$
ε_{ij} i, jth component of the tensor ε
$\mathring{\bar{p}}$ time derivative of $\bar{p}$
$\bar{m}^{klm}$ variable $\bar{m}$ of volume element k, l, m

I. Introduction

Bond graphs are used for modelling in a number of domains of physics. It is possible to represent simple E-M devices with bond graphs (**1**), but the general E-M domain is not yet included. The existing description of E-M devices is based on a few assumptions. Quasi-static conditions, for instance, are assumed to apply. In other words, E-M radiation is neglected. In this paper neither this nor other assumptions, such as homogeneous fields, isotropic linear material etc., are made *a priori*.

The aim of this paper is to describe E-M phenomena together with other domains of physics in one framework (generalized bond graphs, GBG) (**2**, **3**). This may be valuable for the modelling of systems in which the E-M domain, besides other domains, is involved. Also, those studying E-M fields may be attracted to this description based upon one framework, because it provides a way to understand E-M phenomena by means of analogies to phenomena in other domains. The GBG framework is used because it combines thermodynamic theory and the bond graph approach.† The bond graph approach has been shown to be very useful for the modelling of physical systems, while the thermodynamic basis of the GBG concept makes it possible to include the E-M phenomena in a non *ad hoc* way. In Section IV, it is shown that for non moving matter, the E-M phenomena can also be represented by a conventional bond graph. If moving matter is involved the conventional bond graph approach fails to describe the phenomena. The GBG concept, however, makes it possible to describe E-M phenomena in moving matter.

II. Energy Density

Because the bond graph concept is based on power continuity (**1**), it is necessary to discuss the power related to E-M fields. Commonly, the E-M power is described with the Poynting vector Eq. (2). The Poynting vector may be interpreted as the

† The basic idea behind the GBG framework is the decomposition of domains which have two types of storage (mechanics, E-M) into two new domains which only have one type of storage (**3**).

intensity of energy flow at a point in the E-M field; i.e. the energy per second crossing a unit area whose normal is oriented in the direction of the Poynting vector (**4**). Although it is not always stated explicitly, the definition of the Poynting vector is based on a convention. It is possible to define infinitely many power density vectors all of which describe E-M phenomena correctly (**5**). A similar conclusion is valid for the E-M energy density. Equations (1) and (3) provide two possible energy density differentials, whereas the related power density vectors are given by Eqs (2) and (4), i.e.

$$\mathrm{d}W = \bar{E}\cdot\mathrm{d}\bar{D}+\bar{H}\cdot\mathrm{d}\bar{B} \tag{1}$$

$$\bar{S} = \bar{E}*\bar{H} \quad \text{(Poynting vector)} \tag{2}$$

$$\mathrm{d}W = \phi\,\mathrm{d}\rho+\bar{A}\cdot\mathrm{d}\bar{j} \tag{3}$$

$$\bar{S} = \phi\bar{j}. \tag{4}$$

The volume integral of one of the E-M energy density equations (1), (3) over all space is the same for every possible energy density. Hence, the amount of E-M energy is known, but not its distribution in space. The rate of change of the energy density cannot be defined unambiguously either. So the surface integral of an E-M power density vector over a closed surface is not the same for each possible power density vector. Still, neither one of the possible energy densities nor the possible power vectors can be proved to be (in)correct. Therefore, one is free to choose a convenient energy density expression for the modelling of E-M systems. In this paper expression (1) and the related power density vector (Poynting vector) (2) are used. With use of Eq. (1) the E-M energy stored in a small volume can be derived as in (5). The energy flow to this volume is given by Eq. (6), i.e.

$$\mathrm{d}W = (\bar{E}\cdot\mathrm{d}\bar{D}+\bar{H}\cdot\mathrm{d}\bar{B})\,\mathrm{d}V \tag{5}$$

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\oint\!\!\oint_S (\bar{E}*\bar{H})\,\overline{\mathrm{d}s}. \tag{6}$$

Two arguments support the use of Eqs (1) and (2). First, these equations are used in many textbooks (**4–6**). Hence, a bond graph based on these equations is more familiar than a bond graph based on less familiar expressions. A second argument is that expressions (1) and (2) are based on local energy conservation or power continuity (i.e. the assumption that energy can only leave a volume through the surface surrounding it). Because the velocity of light (c) is not assumed to be infinite, local energy conservation is essential for a description with the use of bond graphs. If c is finite and local energy conservation is not assumed then without further extension, the bond graph approach fails to describe the E-M phenomena: in the bond graph concept the energy of a system is stored in its storage elements; there is no energy "on its way".

III. Storage: Electromagnetic State Variables

In the GBG concept the change of the total stored energy W is given by Eq. (7); (**2, 3**). This means that Eq. (5) has to be rewritten in the form of (7) in order to include the

E-M domain. In other words, the electric and magnetic state variables have to be defined. Because E-M fields are not assumed to be homogeneous, volume discretization is applied, i.e. the system is supposed to consist of a number of small cubic volumes. These volumes are considered small enough to assume the fields inside them to be homogeneous. The problem to be discussed may then be reformulated as the description of the electromagnetic interaction of a cube with its surrounding cubes.

$$\mathrm{d}W = \Sigma_j e_j \,\mathrm{d}q_j. \tag{7}$$

First, the E-M state of a small cube is defined. As stated by Breedveld (**2**), the choice of the state variables is more or less subjective because they are the primitives of the GBG theory. It is important to define the E-M state vectors in line with the already used variables for simple lumped electric or magnetic devices (magnetic flux and electric charge). In this way the conventional description of the electric and magnetic domain in bond graphs is a special case of the more general description discussed in this paper. For this reason the electric state vector (vector charge displacement $\bar{Q}$) is defined as given by Eq. (8). The magnetic state vector, the vector flux $\bar{\Phi}$, is provided by Eq. (9), where L is the side of the cube. These state variables are discussed in an intuitive way in the Appendix.

$$\bar{Q} = L^2\bar{D} \tag{8}$$

$$\bar{\Phi} = L^2\bar{B}. \tag{9}$$

According to the GBG concept, the state of a physical domain is given by its extensive variable q_j. The related intensive variable (effort) e_j can be found as the partial derivative of the energy W with respect to q_j (10). With use of this equation, the magnetic effort ($\bar{m}$, vector magneto motive force) and electric effort ($\bar{U}$, vector voltage) can be derived as in (11) and (12). Equations (11) and (12) are also discussed in the Appendix.

$$e_j = \frac{\partial W}{\partial q_j} \tag{10}$$

$$\bar{U} = L\bar{E} \tag{11}$$

$$\bar{m} = L\bar{H}. \tag{12}$$

With the use of the generalized magnetic and electric state and effort vectors, the change of energy stored in a small cube is provided by Eq. (13). This equation has the form of (7), which indicates that the inclusion of E-M phenomena in the GBG framework is possible. The E-M energy storage and dissipation are represented by three multibond graph elements (Fig. 1). [The notation of the multibond graphs in this paper is in line with Refs. (**7, 8**).]

$$\mathrm{d}W = \bar{m}\cdot \mathrm{d}\bar{\Phi} + \bar{U}\cdot \mathrm{d}\bar{Q}. \tag{13}$$

The constitutive relations of the multiport C and R elements in Fig. 1 can be derived with use of (8), (9), (11) and (12), resulting in (14)–(22). For reasons of simplicity the extensive variables are expressed as a function of the intensive

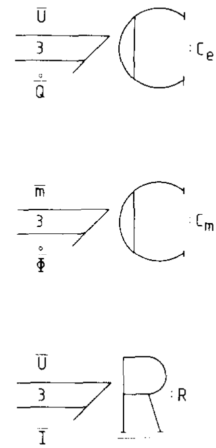

FIG. 1. Multibond graph elements or multiport elements representing storage and dissipation of E-M energy. C_E is the electric storage element, C_M is the magnetic storage element and R is the dissipator.

variables. The Maxwell reciprocity relations hold for these equations (i.e. $u_{ij}L = u_{ji}L$ and $\varepsilon_{ij}L = \varepsilon_{ji}L$, $\forall\, i, j$).

$$\mathrm{d}Q_1 = L\varepsilon_{11}\,\mathrm{d}U_1 + L\varepsilon_{12}\,\mathrm{d}U_2 + L\varepsilon_{13}\,\mathrm{d}U_3 \tag{14}$$

$$\mathrm{d}Q_2 = L\varepsilon_{21}\,\mathrm{d}U_1 + L\varepsilon_{22}\,\mathrm{d}U_2 + L\varepsilon_{23}\,\mathrm{d}U_3 \tag{15}$$

$$\mathrm{d}Q_3 = L\varepsilon_{31}\,\mathrm{d}U_1 + L\varepsilon_{32}\,\mathrm{d}U_2 + L\varepsilon_{33}\,\mathrm{d}U_3 \tag{16}$$

$$\mathrm{d}\Phi_1 = L\mu_{11}\,\mathrm{d}m_1 + L\mu_{12}\,\mathrm{d}m_2 + L\mu_{13}\,\mathrm{d}m_3 \tag{17}$$

$$\mathrm{d}\Phi_2 = L\mu_{21}\,\mathrm{d}m_1 + L\mu_{22}\,\mathrm{d}m_2 + L\mu_{23}\,\mathrm{d}m_3 \tag{18}$$

$$\mathrm{d}\Phi_3 = L\mu_{31}\,\mathrm{d}m_1 + L\mu_{32}\,\mathrm{d}m_2 + L\mu_{33}\,\mathrm{d}m_3 \tag{19}$$

$$U_1 = I_1/L\sigma_1 \tag{20}$$

$$U_2 = I_2/L\sigma_2 \tag{21}$$

$$U_3 = I_3/L\sigma_3. \tag{22}$$

IV. Maxwell's Equations

Generalized bond graphs are based on a synthesis of (irreversible) thermodynamics (description of nonlinear storage elements and dissipators) and network methods (the interconnection structure) (**3**). The storage elements and dissipator of the E-M domain have been treated in the previous section. In this section, the interconnection of these elements is discussed. The interconnection is described with a junction structure.

E-M phenomena are governed by Maxwell's equations; in fact, it is the interconnection of the storage elements and dissipators which is described by these

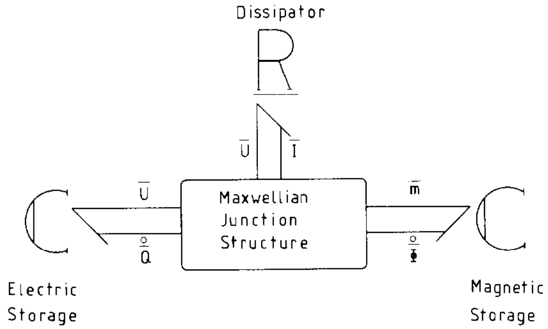

FIG. 2. Word bond graph of an E-M system.

equations. Maxwell's equations (23) and (24)† couple the electric field intensity ($\bar{E}$) to the magnetic induction ($\bar{B}$), and the magnetic field intensity ($\bar{H}$) to the electric displacement and current density ($\bar{D}, \bar{j}$).

$$-\frac{\mathrm{d}\bar{B}}{\mathrm{d}t} = \operatorname{rot} \bar{E} \tag{23}$$

$$\bar{j} + \frac{\mathrm{d}\bar{D}}{\mathrm{d}t} = \operatorname{rot} \bar{H}. \tag{24}$$

The problem to be discussed in this section may be reformulated as the description of Maxwell's equations by means of a junction structure. The word bond graph of Fig. 2 shows the junction structure together with the magnetic and electric storage elements and the dissipator. The structure of the bond graph of an E-M system (Fig. 2) has similarities with the structure of the bond graph of a mechanical system (Fig. 3) (**2**). In both cases, a junction structure, representing the governing equations of the domain (Maxwell, Newton), connects the two storage elements in the dual domains.

An example of such a junction structure is provided by the bond graph of the spring–mass system in Fig. 4. The bond graph shows the two storage elements, potential and kinetic, and the connecting junction structure, the gyrator. The gyrator, called symplectic gyrator (**3**), represents Newton's second law of motion Eq. (25) and expression (26) which is valid in a Lagrangian frame of reference, i.e.

$$\bar{F} = \frac{\mathrm{d}\bar{p}}{\mathrm{d}t} \tag{25}$$

$$\bar{v} = \frac{\mathrm{d}\bar{x}}{\mathrm{d}t}. \tag{26}$$

† Frequently two more equations are included as part of Maxwell's system (i.e. $\nabla \cdot \bar{B} = 0$, $\nabla \cdot \bar{D} = \rho$). It must be noted, however, that if the conservation of charge is assumed, these are not independent relations (**4**), hence, Eqs (23) and (24) are sufficient.

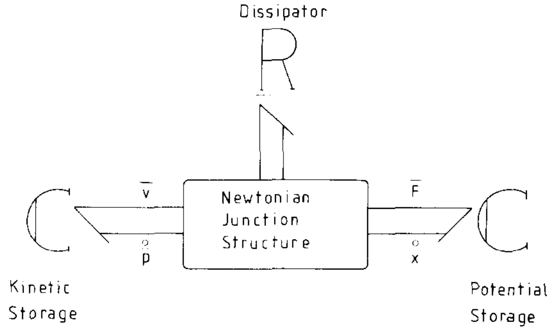

FIG. 3. Word bond graph of a mechanical system.

The junction structure (or generalized network) of Fig. 2 represents all E-M energy flows between the elements, including radiation (quasi-static conditions are not assumed; the E-M radiation is not neglected). This may be confusing because for the description of an electric network it is always assumed that the E-M radiation is negligible, but here a network is discussed which represents E-M radiation. This paradox becomes clearer if the assumption of electric circuit theory, that quasi-static conditions apply, is reformulated as follows: it is assumed that no other energy exchanges take place except those described by the network. For an electric network this means that the energy exchange between the elements is assumed to take place exclusively through the wires connecting the elements of the system with each other (no radiation). In the (generalized) network description of E-M phenomena presented here a similar assumption is made. It is assumed that there is no other way of transporting energy within the system than by means of E-M radiation and electric currents. It is emphasized that because the electric network assumption that all energy exchanges take place through wires, is not made it will become possible to represent E-M radiation in a (generalized) network. The aim of this section is to find this generalized junction structure.

With the use of Maxwell's equation (23), the time rate of the magnetic state vector (flow) can be expressed in terms of the curl of the electric effort (27). Equation (24)

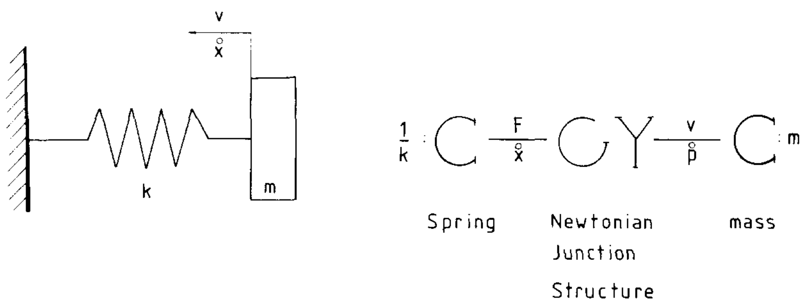

FIG. 4. One-dimensional spring–mass system and its generalized bond graph.

can also be rewritten with the use of the variables defined in Section III (28). Equations (27) and (28) show that the junction structure, which is to be derived, couples the electric effort to the magnetic flow and the magnetic effort to the electric flow, i.e.

$$\begin{aligned} L^2 \operatorname{rot} \bar{E} &= -\frac{\mathrm{d}}{\mathrm{d}t} L^2 \bar{B} \\ L \operatorname{rot} \bar{U} &= -\frac{\mathrm{d}}{\mathrm{d}t} \bar{\Phi} \end{aligned} \tag{27}$$

$$\begin{aligned} L^2 \operatorname{rot} \bar{H} &= \frac{\mathrm{d}}{\mathrm{d}t} L^2 \bar{D} + L^2 \bar{j} \\ L \operatorname{rot} \bar{m} &= \frac{\mathrm{d}\bar{Q}}{\mathrm{d}t} + \bar{I}. \end{aligned} \tag{28}$$

For the description of the E-M storage the volume has been discretized (Section III). It was assumed that the volumes are small enough to consider the fields inside them as homogeneous. In Fig. 5 a number of these volume elements is shown and an expression for rot $\bar{U}$ and rot $\bar{m}$ can be found. For example, Eq. (29) provides the component of the rotation of vector field $\bar{G}$ directed parallel with the "1" axis. In a similar way (30) and (31) are derived.

$$[\operatorname{rot} \bar{G}]_1 = \frac{1}{2L}(G_3^{kl+1m} - G_2^{klm+1} + G_2^{klm-1} - G_3^{kl-1m}). \tag{29}$$

With the use of the electric and magnetic state and effort vectors (23) and (24) can be approximated with first order accuracy by (30) and (31). In these equations, the flow of volume element (k, l, m) is related to the effort of its neighbours $(k-1, l, m)$,

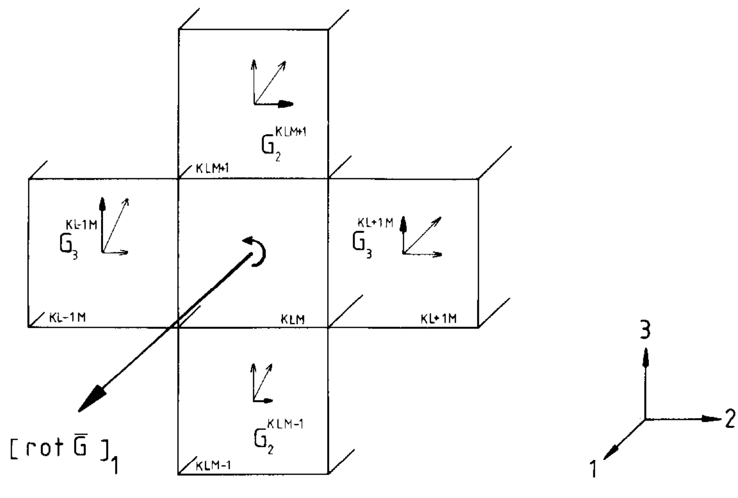

FIG. 5. Vector field $\bar{G}$.

$(k+1, l, m)$, etc.

$$-\frac{d\bar{\Phi}^{klm}}{dt} = \begin{bmatrix} U_3^{klm} - U_2^{klm} + U_2^{klm-1} - U_3^{kl-1m} \\ U_1^{klm} - U_3^{klm} + U_3^{k-1lm} - U_1^{klm-1} \\ U_2^{klm} - U_1^{klm} + U_1^{kl-1m} - U_2^{k-1lm} \end{bmatrix} \tag{30}$$

$$\bar{I}^{klm} + \frac{d\bar{Q}^{klm}}{dt} = \begin{bmatrix} m_3^{klm} - m_2^{klm} + m_2^{klm-1} - m_3^{kl-1m} \\ m_1^{klm} - m_3^{klm} + m_3^{k-1lm} - m_1^{klm-1} \\ m_2^{klm} - m_1^{klm} + m_1^{kl-1m} - m_2^{k-1lm} \end{bmatrix}. \tag{31}$$

Equations (30) and (31) show that it is possible to represent Maxwell's equations (23) and (24) by a multiport gyrator (3). This element relates the electric flow to the magnetic effort, and the magnetic flow to the electric effort. The constitutive relation of the gyrator is an extension of (30) and (31), discussed in more detail in (9).

With the use of the multiport gyrator, representing Maxwell's equations, and the storage elements and dissipator of Section III, the bond graph of an E-M system can be completed. Figure 6 shows the bond graph of a system with n volume elements. Each volume element has an electric and a magnetic storage element and a dissipator. These 3 n elements are coupled by the multiport gyrator, representing Maxwell's equations; Fig. 7 shows the same bond graph in a more compact notation (7).

The junction structure representing Maxwell's equations is more complicated than the one of Fig. 4 [representing Newton's second law of motion and the identity (26)]. This is not unexpected: the governing equations of E-M phenomena

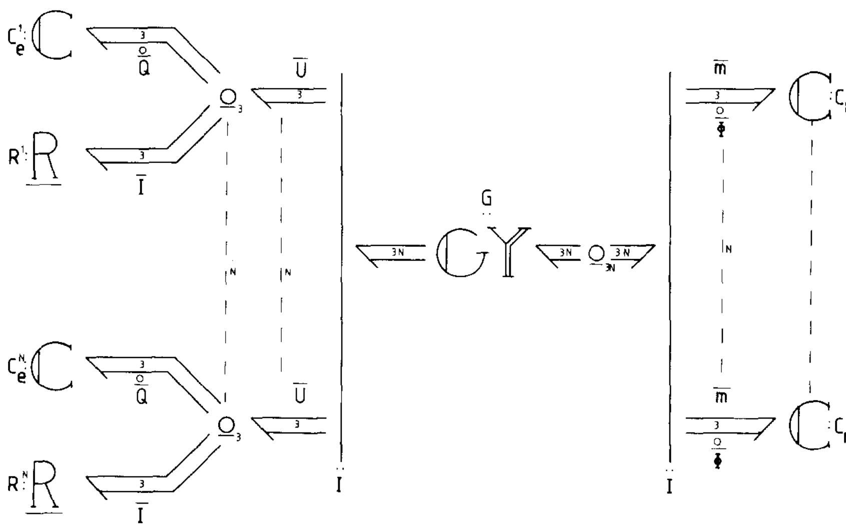

FIG. 6. Multibond graph of an E-M system with n elements. C_E is the electric storage element, C_M is the magnetic storage element and R is the dissipator.

(Maxwell) are more complicated than the governing equations of mechanics (Newton).

For a mechanical system the ordinary bond graph is obtained by combining the kinetic storage element with the Newtonian symplectic gyrator. This is possible because the gyrator is non-essential, i.e. it can be eliminated (**2**, **10**, **11**). The same applies to an E-M system, because the Maxwellian gyrator (Fig. 6) is also non-essential. In Fig. 8, the resulting ordinary bond graph of a cubic volume element is shown. The storage of electric (C-element) and magnetic energy (I-element) are shown, as well as the E-M radiation, to be the bonds connecting the bond graph of the cube with the bond graph of the neighbouring cube. The dissipation is omitted. The bonds which represent the E-M radiation do not have the magnetic or electric conjugated variables, but rather have $\bar{U}$ and $\bar{m}$ as conjugated variables. That it is possible to represent E-M power with an expression which includes only one electric and one magnetic variable, can be shown by the use of (32). The four variables of Eq. (32) are not independent, due to Maxwell's equations: $\bar{m}$ is coupled to $\bar{Q}$, and $\bar{U}$ is coupled to $\bar{\Phi}$. Hence, $\bar{U}$ and $\bar{m}$ may be used as conjugated variables to represent E-M power. It is not possible to use another pair of variables (for instance $\bar{U}$ and $\mathrm{d}\bar{Q}/\mathrm{d}t$), because the E-M flows ($\mathrm{d}\bar{Q}/\mathrm{d}t$ and $\mathrm{d}\bar{\Phi}/\mathrm{d}t$) are expressed as a function of a derivative with respect to position (the rotation operator) of the efforts ($\bar{U}$ and $\bar{m}$). In other words, the gyrator representing Maxwell's equation has a fixed causality, so with the use of the efforts the flows can be found, but with the flows the efforts are not completely determined. The usual definition of the E-M power density vector, the Poynting vector, is related to the representation with use of $\bar{U}$ and $\bar{m}$; it is defined as the vector product of $\bar{E}$ [related to the electric effort $\bar{U}$, (8)] and $\bar{H}$ [related to the magnetic effort $\bar{m}$, (9)]. The energy flow from cube $kl-1m$ to cube klm (Fig. 8), for instance, is represented by S_{20}, which is the normal component of the Poynting vector on the surface of the cube times the area of this surface. The Poynting vector is immediately shown by the bond graph of Fig. 8 as the difference between the powers represented by the two bonds which connect cube $kl-1m$ with its neighbour klm, as expressed by (33).

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \bar{U}\cdot\frac{\mathrm{d}\bar{Q}}{\mathrm{d}t} + \bar{m}\cdot\frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}t} \tag{32}$$

$$S_{20} = \bar{S}\cdot\bar{n}L^2 = U_3m_1 - U_1m_3 = L^2(E_3H_1 - E_1H_3). \tag{33}$$

The junctions in the bond graph of Fig. 8 represent Maxwell's equations. The

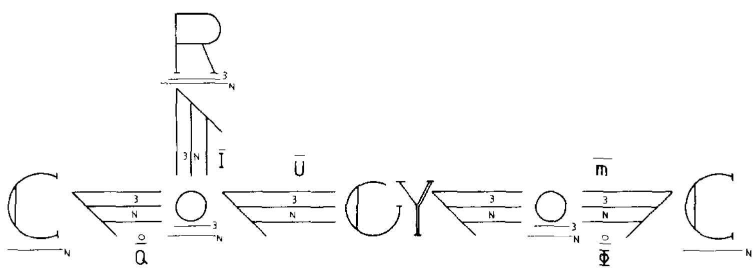

FIG. 7. Multibond graph of Fig. 6 in compact notation.

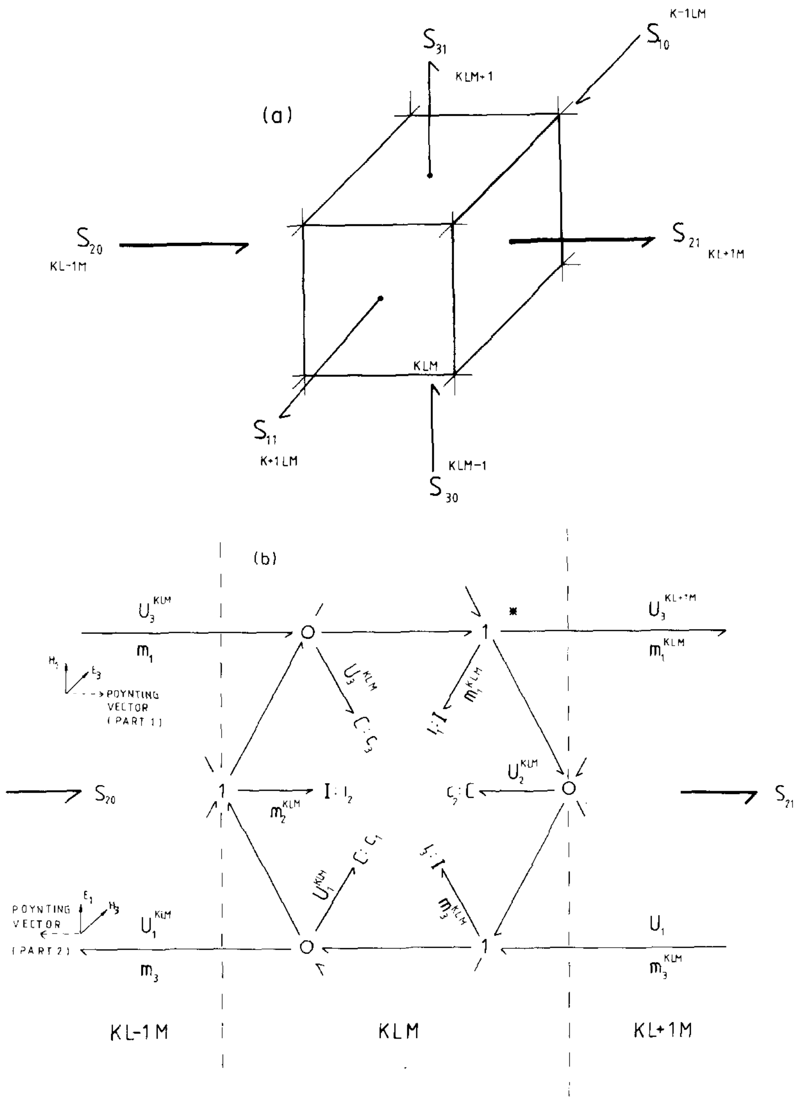

FIG. 8. Cube with six power vectors related to its lateral surfaces. For simplicity only S_{20} and S_{21} are presented in the bond graph.

constitutive relation of the marked 1-junction in Fig. 8 for instance, provided by Eq. (34), is similar to the first line of (30) (Maxwell's equations in a first order approximation),

$$-\frac{\mathrm{d}\Phi_1}{\mathrm{d}t} = U_3^{klm} - U_2^{klm} + U_2^{klm-1} - U_3^{kl-1m}. \tag{34}$$

Using the bond graph of Fig. 8 an E-M device can be modelled by volume discretization; every volume element is represented by a bond graph similar to the one in Fig. 8. As an example, a model of the waveguide of Fig. 9 is shown. The cross-section of the waveguide is divided in four sections. In Fig. 9, the bond graph of such a cross-section is shown. The dissipators represent the conductivity of the sides of the waveguide.

It is possible to include those parts of an E-M system described by lumped elements; for instance, the characteristic impedance of a coaxial cable, in the model. However, all volume elements of the parts of the system which are not described by lumped elements and in which the fields are not negligible, have to be included. In

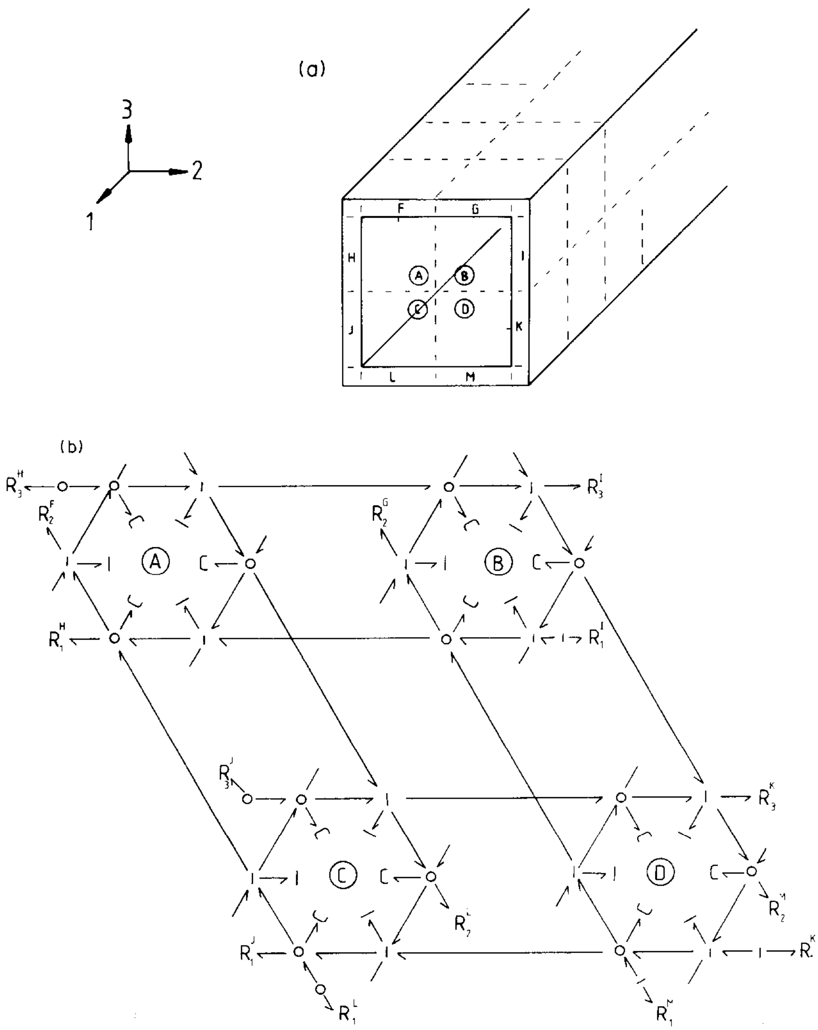

FIG. 9. (a) Wave guide. (b) Bond graph for a cross-section of the waveguide.

other words, only radiation within the modelled device is included in the model. The description of the waveguide of Fig. 9, for instance, is based on the assumption that the E-M fields outside the waveguide are zero. Radiation to or from other devices can be included in the description in a number of ways, for instance, by introduction of a lumped connecting element. It is, of course, also possible to include all involved volume elements into one system.

V. E-M Forces

In the previous sections the inclusion of the E-M domain in the GBG framework has been discussed. It was shown that it is also possible to describe an E-M system with respect to a matter frame with conventional bond graphs. As stated in the Introduction, bond graphs are especially useful for the modelling of systems in which several physical domains are involved. In this and the next section it is demonstrated that the description of a system which also involves another domain, is possible in a direct way. As an example, a bond graph will be derived which models an E-M system and its interaction with the mechanical domain. The GBG approach is essential for this problem; it is not possible to describe the phenomena with conventional bond graphs.

In the previous sections, the E-M behaviour is described in a frame stationary with respect to the matter, a Lagrangian frame. This means that forces, due to the E-M field and exerted on the matter, are not represented in the model. In order to include these E-M forces, the fields have to be evaluated with respect to a Eulerian frame of reference.

Consider a control volume stationary with respect to the observer frame. Through this control volume incompressible homogeneous matter flows with a velocity $\bar{v}$ with respect to the observer frame. Various extensive variables (entropy, mechanical momentum etc.) are stored in the matter which temporarily resides in the control volume. In order to model the convective phenomena in the control volume the rates of these extensive variables (flows) have to be known. The rate of a directed (vector) extensive variable of the control volume is in general form given by

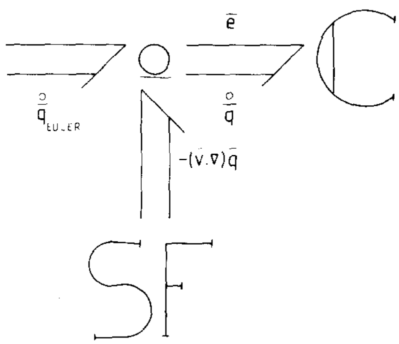

FIG. 10. Transformation from Lagrangian to Eulerian frame of reference.

Eq. (35). An example of this "material derivative" is the hydrodynamic case in which a fluid exerts a pressure in a Eulerian frame due to the momentum $\bar{p}$ which is convected with a velocity $\bar{v}$ with respect to the Eulerian frame (36) (**5**). This transformation from a representation with respect to a Lagrangian frame to a representation with respect to a Eulerian frame is shown by the multibond graph of Fig. 10. The source in this bond graph is a provisional correction term for the flow and the associated power resulting from the frame transformation.

$$\frac{d\bar{q}}{dt}_{\text{Euler}} = \frac{d\bar{q}}{dt}_{\text{Lagrange}} + (\bar{v}\cdot\nabla)\bar{q} \tag{35}$$

$$-\nabla P = \bar{F}_{\text{Euler}} = \mathring{\bar{p}}_{\text{Euler}} = \mathring{\bar{p}}_{\text{Lagrange}} + (\bar{v}\cdot\nabla)\bar{p} \tag{36}$$

In the previous sections, the E-M phenomena have been included in the GBG framework and are described in the same way as in more common domains of physics. The transformation from Lagrangian to Eulerian frame can also be applied to the E-M domain. For a magnetic system this results in the bond graph of Fig. 11. For an electric system the transformation is more complicated (Fig. 12). Not only has the stored extensive variable $\bar{Q}$ to be taken into consideration, the current is also dependent on the velocity of the observer frame. Charge stationary with respect to the matter is moving with respect to the observer frame; hence, the currents with respect to the matter frame and the observer frame are generally not equal. The source on the left-hand side represents this convection of charge.

Figures 11 and 12 represent bond graphs for the magnetic and electric storage and dissipation with respect to a Eulerian frame. As Maxwell's equations are independent of the reference frame (**4**) the multibond graph for an E-M system which moves with respect to a Eulerian frame can be found. Combining Fig. 11 and Fig. 12 with the multiport gyrator, representing Maxwell's equations, yields the bond graph of Fig. 13. The three flow sources in this graph represent the power exchange between the mechanical part of the system and the E-M domain. Replacement of source 2 to the right-hand side of the gyrator and of source 3 to the left-hand side of the gyrator yields the bond graph of Fig. 14. Finally, after substitution of the three sources by bonds with the kinetic domain the bond graph of Fig. 15 is obtained. This

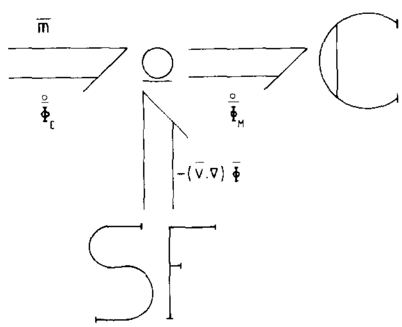

FIG. 11. Transformation from Lagrangian to Eulerian frame of reference (E, Eulerian frame, M, matter frame).

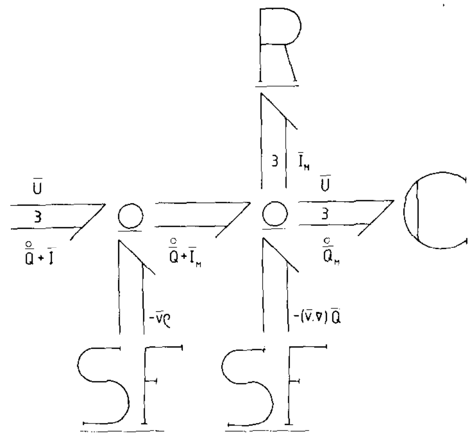

FIG. 12. Electric storage element and dissipator transformed to the Eulerian frame (M matter frame).

multibond graph represents the E-M behaviour including the E-M forces. The force expressions (Lorentz force etc.) are contained in the bond graph at the kinetic 0-junction. Equation (37) shows the force expression. With the use of (38) and the definition of $\bar{Q}$, $\bar{I}$ and $\bar{\Phi}$ (8), (9) and (11), the E-M force can be expressed in a continuous form (39), i.e.

$$\bar{F} = L^2\rho\bar{U} - \frac{\bar{\Phi}}{L}*\left(\bar{I}+\frac{\mathrm{d}\bar{Q}}{\mathrm{d}t}\right)+\frac{\bar{Q}}{L}*\frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}t} \tag{37}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\bar{Q}*\bar{\Phi}) = \frac{\mathrm{d}\bar{Q}}{\mathrm{d}t}*\bar{\Phi}+\bar{Q}*\frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}t} \tag{38}$$

$$\bar{F} = \int_V \left[\rho\bar{E}+\bar{j}*\bar{B}+\frac{\mathrm{d}}{\mathrm{d}t}(\bar{D}*\bar{B})\right]\mathrm{d}v. \tag{39}$$

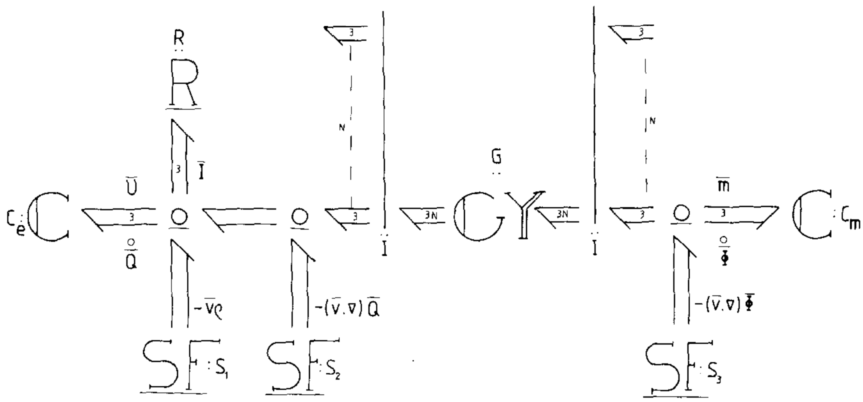

FIG. 13. E-M system with respect to Eulerian frame.

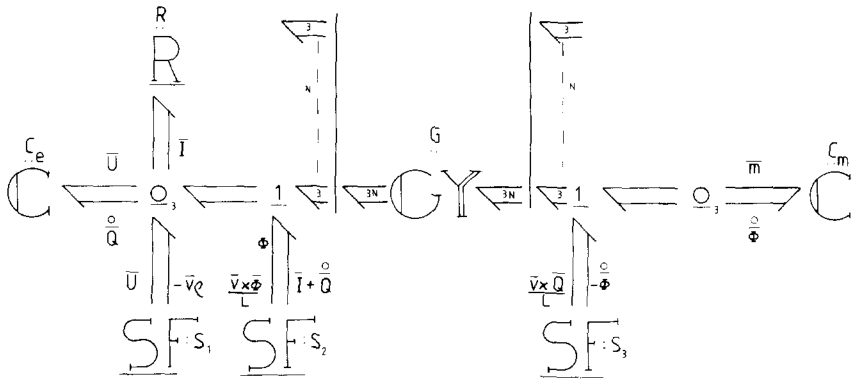

FIG. 14. E-M system with respect to Eulerian frame.

The first two terms of (39) are rather well-known: the first one represents the force on a charge density due to an electric field, the second term represents the Lorentz force. It is interesting to note that the bond graph provides the last term of (39), which is commonly not known and usually left out, because it is very small. In order to find this term in the conventional way (4) a long derivation is required. It must be noted, however, that Stratton's derivation (4) provides an equation with more terms, (40), but even this equation is incomplete (12)

$$\bar{F} = \int_V \left[\rho\bar{E} + \bar{j} * \bar{B} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{D} * \bar{B} - \frac{\bar{E} * \bar{H}}{c^2} \right) - E^2\nabla\varepsilon - H^2\nabla\mu \right] \mathrm{d}V. \tag{40}$$

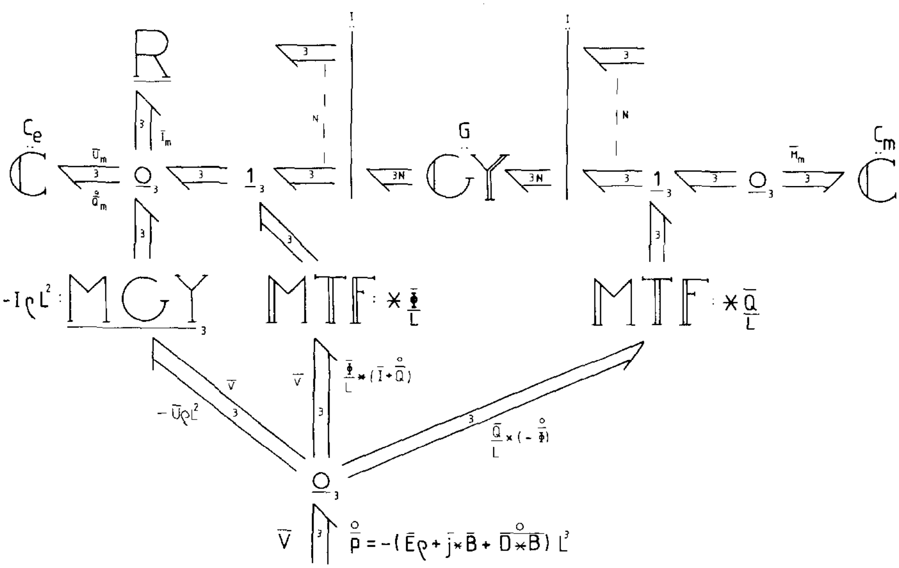

FIG. 15. E-M system with kinetic domain.

In this paper, the force equation is restricted to the terms of (39). In other words, the last three terms of (40) are left out. In fact, these terms are far too small to be easily detected. Equation (37) shows that the transformer on the right-hand side of Fig. 15 also adds a negligible term to the force expression and therefore may be omitted from the bond graph.

It was shown in Section IV that the gyrator representing Maxwell's equations is not essential (3) for a system with non moving matter. If an E-M system is not described with respect to a frame stationary to the matter then it is not possible to eliminate the Maxwellian gyrator; the description of E-M phenomena with respect to a Eulerian frame makes the gyrator essential. As in the case of a mechanical convection (3) it is not possible to describe the E-M phenomena in moving matter with the use of conventional bond graphs. The GBG approach makes the description possible.

VI. Conclusion

In the GBG framework generalized E-M state and effort variables can be defined, which make it possible to describe moving E-M systems, including the resulting E-M forces, with multibond graphs. Field transformations due to the velocity of the moving matter are also described by the bond graph; the author is preparing a paper on this subject. A number of assumptions which are often used to describe E-M phenomena such as: isotropic and linear fields, periodic time functions, quasi-static conditions, etc. are not necessary. Using the material derivative (35) and Maxwell's equations it is possible to derive the E-M forces and the field transformations in a direct way. The bond graph shows that, for instance, hydraulic "velocity pressure" and the forces due to a moving E-M field are both based on the transformation from a Lagrangian frame to a Eulerian frame. It demonstrates one of the important advantages of bond graphs that unknown domains can be dealt with by utilizing analogies to domains that one understands better. The strength of the generalized bond graph approach is confirmed by the structured way in which the general E-M domain can be introduced in the GBG framework.

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Appendix

In this appendix the defined electric and magnetic state variables are discussed more intuitively.

The vector flux $\bar{\Phi}$ of a cubic volume element is defined by Eq. (9), i.e.

$$\bar{\Phi} = L^2\bar{B}. \tag{9}$$

It can be interpreted as the magnetic flux through the cube, i.e. the component of $\bar{\Phi}$ parallel to the x-axis is equal to the surface area of the side of the cube perpendicular to the x-axis, times the x-component of the magnetic induction. A similar statement can be given for the y and z-component. Hence the vector flux may be described as the magnetic flux through the cube. It may be clear that the same statement is valid for the vector charge displacement $\bar{Q}$; its x-component is the charge displaced through the cube in the direction of the x-axis. For the vector current $\bar{I}$ similar arguments hold. The vector voltage $\bar{U}$ of the cube of Fig. 16 is a combination of the voltages which can be measured over a cube. Its first component is the voltage between the parallel sides of the cube which are perpendicular to the x-axis. The y and z-component can be discussed in a similar way. Although it is not very common to speak of the magnetomotive force between two points, it can be defined similarly to the vector voltage.

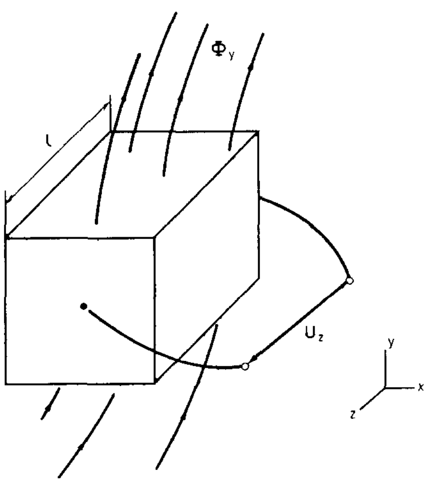

FIG. 16. Cubic volume element.